SECTION 15.6: TANGENT PLANES AND LINEARIZATION

RECALL: If $f'(a)$ exists, then the **tangent line** to the graph of $y = f(x)$ at $(a, f(a))$ is:

$$y = f(a) + f'(a)(x - a)$$

or

$$y = f(a) + (\text{rate of change of } f \text{ with respect to } x) \cdot (\text{change in } x)$$

DEFINITION: If $f_x(a, b)$ and $f_y(a, b)$ exist, we formally define the **tangent plane** at $(a, b, f(a, b))$:

$$z = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

or

$$z = f(a, b) + (\text{rate of change of } f \text{ with respect to } x) \cdot (\text{change in } x) + (\text{rate of change of } f \text{ with respect to } y) \cdot (\text{change in } y)$$

JUSTIFICATION:

- The tangent line from the slice $y = b$ at $(a, b, f(a, b))$ is:
- Direction vector for your tangent line:
- The tangent line from the slice $x = a$ at $(a, b, f(a, b))$ is:
- Direction vector for your tangent line:
- How can we get a vector normal to both of these tangent vectors?
- Find the equation of the plane containing $(a, b, f(a, b))$ that has the normal vector you just found.

EXAMPLE 1: Find the equation of the tangent plane at the indicated point.

Graph the function along with the tangent plane near the point of tangency. What do you notice?

1. $f(x,y) = x^2 - 3xy + y^3$ at $(x,y) = (-1,1)$.

Ans: $z = -5x + 6y - 6$

2. $f(x,y) = e^{2x+y}\cos(y)$ at $(x,y) = (0,0)$.

Ans: $z = 2x + y + 1$

3. $f(x,y) = \ln\left(x + y^2\right)$ at $(x,y) = (0,-1)$.

Ans: $z = x - 2y - 2$

EXAMPLE 2: Let:

$$f(x,y) = \begin{cases} \dfrac{xy^2}{x^2+y^4} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

1. Determine: $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ along the paths $x = y$ and $x = y^2$. Does $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ exist?

   Ans: along $x = y$ the limit is 0; along $x = y^2$, the limit is $\dfrac{1}{2}$. Hence, $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ does not exist.

2. Use the limit definition of $f_x(0,0) = \lim\limits_{h\to 0} \dfrac{f(h,0) - f(0,0)}{h}$ to show $f_x(0,0) = 0$.

3. Use the limit definition of $f_y(0,0) = \lim\limits_{k\to 0} \dfrac{f(0,k) - f(0,0)}{k}$ to show $f_y(0,0) = 0$.

4. Write the equation of the tangent plane to $z = f(x,y)$ at $(x,y) = (0,0)$. Graph the function and the tangent plane near the origin. Does the tangent plane do a reasonable job approximating the surface?

DIFFERENTIABILITY AS LOCAL LINEARITY

The last example illustrates that even though a tangent plane **exists**, it may not be good at approximating the surface near the point of tangency. If we want to truly extend what it means to be 'differentiable' we need to extend the notion of '**local linearity**.'

RECALL: If f is differentiable at $x = a$, then f is **locally linear** 'near' $x = a$. That is, 'near' $x = a$,

$$f(x) \approx f(a) + f'(a)(x - a) = \text{the tangent line at } (a, f(a))$$

You may recall that the **linearization** of f at $x = a$ is then $L_a(x) = f(a) + f'(a)(x - a)$.

More formally, we have the following theorem which was used in Calculus 1 to prove all the derivative properties:

LOCAL LINEARITY THEOREM: If f is differentiable at $x = a$, then there is a function $\epsilon(x)$ such that:

$$f(x) = f(a) + f'(a)(x - a) + \epsilon(x)(x - a),$$

where $\epsilon(x) \to 0$ as $x \to a$.

The function $\epsilon(x)$ is a type of 'error' between the function $f(x)$ and the linearization $L_a(x) = f(a) + f'(a)(x - a)$.

See me if you've never seen this before and would like more information on this result (I have handouts!)

We generalize these concepts in order to **define** differentiability of functions of several variables.

DEFINITION: f is said to be **differentiable** at (a, b) if there are functions $\epsilon_1(x, y)$ and $\epsilon_2(x, y)$ so that

$$f(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b) + \epsilon_1(x, y)(x - a) + \epsilon_2(x, y)(y - b),$$

where $\epsilon_1(x, y) \to 0$ and $\epsilon_2(x, y) \to 0$ as $(x, y) \to (a, b)$.

Sometimes, it's advantageous to relabel $x - a = \Delta x$ and $y - b = \Delta y$ so $x = a + \Delta x$ and $y = b + \Delta y$.

Doing so, we can rewrite the above condition for differentiability as:

$$f(a + \Delta x, b + \Delta y) = f(a, b) + f_x(a, b)\Delta x + f_y(a, b)\Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y,$$

where $\epsilon_1 \to 0$ and $\epsilon_2 \to 0$ as $\Delta x \to 0$ and $\Delta y \to 0$.

DEFINITION; If $f_x(a, b)$ and $f_y(a, b)$ exist at (a, b), the **linearization** of f at (a, b) is

$$L_{(a,b)}(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b).$$

LOCAL LINEARITY: f is differentiable at (a, b) means $f(x, y) \approx L_{(a,b)}(x, y)$ for points (x, y) near (a, b).

The definition of differentiability can be awkward to work with but we will see its usefulness to help prove properties of differentiable functions. For example:

THEOREM: If f is differentiable at (a, b) then f is continuous at (a, b).

The following theorem gives some easier conditions to check for differentiability.

THEOREM: If f_x and f_y are **continuous** in an open disk containing (a, b), then f is differentiable at (a, b).

EXAMPLE 3: Let $f(x, y) = xe^{x-2y}$.

1. Find $f_x(x, y)$ and $f_y(x, y)$ and use these to prove f is differentiable at all points (x, y) in the plane.

Ans: $f_x(x, y) = e^{x-2y} + xe^{x-2y}$, $f_y(x, y) = -2xe^{x-2y}$. Both are continuous everywhere.

2. Find the linearization of f at $(1, 0.5)$, $L_{(1,0.5)}(x, y)$.

Ans: $L_{(1,0.5)}(x, y) = 2x - 2y$

3. Use $L_{(1,0.5)}$ to approximate $f(0.90, 0.55)$, Compare your answer to the actual retail value of $f(0.90, 0.55)$.

Ans: $L_{(1,0.5)}(0.90, 0.55) = 0.70$. $f(0.90, 0.55) \approx 0.7368$

DIFFERENTIALS

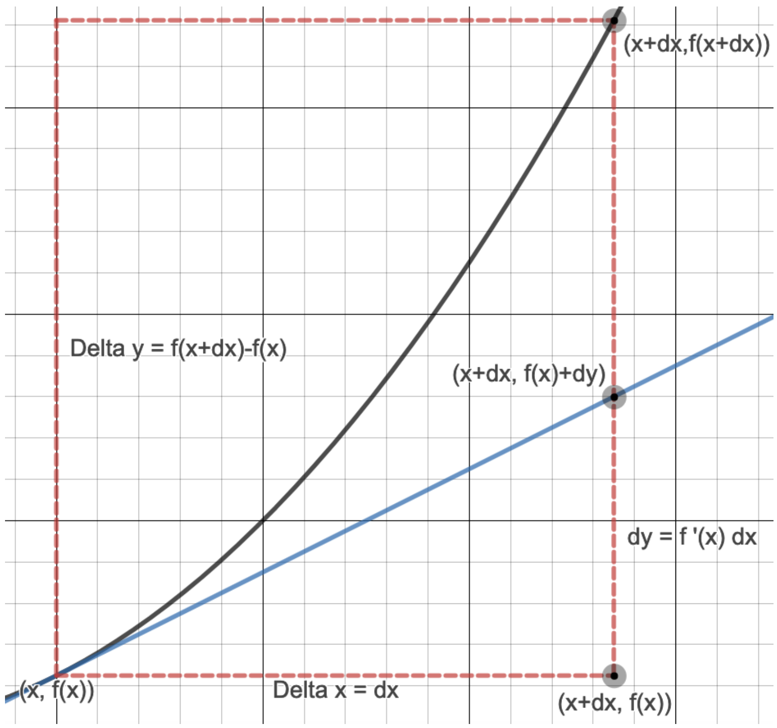
RECALL: If f is differentiable at $x = a$, then near $x = a$, $f(x) \approx f(a) + f'(a)(x - a)$.

Setting $\Delta x = x - a$, so $x = a + \Delta x$, we have $f(x) = f(a + \Delta x) \approx f(a) + f'(a)\Delta x$.

This lead us to the following definition in Calculus 1:

DEFINITION: If $y = f(x)$, the **differential** of x is $dx = \Delta x$. The **differential** of y is $dy = f'(x)\,dx$.

Recall that $\Delta x = dx$ is the change in **inputs** to the function whereas $\Delta y = f(x+\Delta x) - f(x)$ is the corresponding change in **outputs** from the function. The quantity dy is the change in **outputs** on the tangent line. If f is differentiable, then when Δx is small, $\Delta y \approx dy$ since the graph of $y = f(x)$ is approximated by the tangent line.



The long and short of these definitions is the following:

APPROXIMATION USING DIFFERENTIALS: If f is differentiable, then:

$$f(x + \Delta x) \approx f(x) + dy \qquad \text{or} \qquad \Delta y = f(x + \Delta x) - f(x) \approx dy$$

Generalizing to functions of two variables, we have the following.

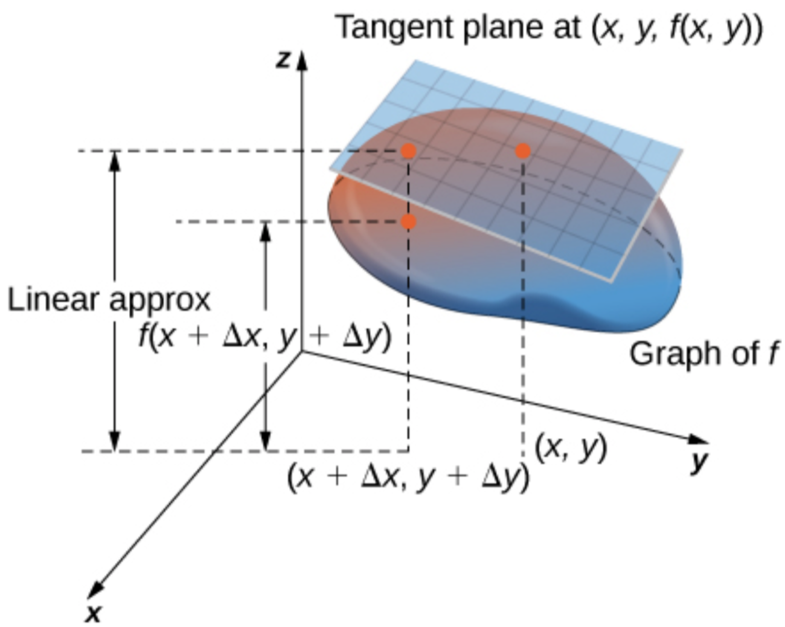
DEFINITION: If $f(x, y)$ is differentiable, then setting $dx = \Delta x$ and $dy = \Delta y$, we define the **total differential**

$$dz = f_x(x, y)\, dx + f_y(x, y)\, dy$$

APPROXIMATION USING DIFFERENTIALS: If f is differentiable, then:

$$f(x + \Delta x, y + \Delta y) \approx f(x, y) + dz \qquad \text{or} \qquad \Delta z = f(x + \Delta x, y + \Delta y) - f(x, y) \approx dz$$

Geometrically, we are approximating the difference in function values on the **surface** $z = f(x, y)$ by the difference in values on the **tangent plane**.



EXAMPLE 4: Let $f(x, y) = xe^{x-2y}$.

1. Find an expression for the total differential, dz for $(x, y) = (1, 0.5)$.

Ans: $dz = 2dx - 2dy$.

2. Find $f(1, 0.5)$ and use differentials to approximate $f(0.90, 0.55)$. Why does your answer look familiar?

Ans: $f(1, 0.5) = 1$. $dx = 0.90 - 1 = -0.10$, $dy = 0.55 - 0.5 = 0.05$ so $dz = 2(-0.10) - 2(0.05) = -0.30$.

Hence, $f(0.90, 0.55) \approx 1 + dz = 1 - 0.30 = 0.70$.

EXAMPLE 5: The density of a substance, ρ, is calculated by dividing its mass, m, by its volume, V: $\rho = \dfrac{m}{V}$.

A scientist collects 5 ± 0.5 mL of a substance and determines its mass to be 68.2 ± 0.1g.

She computes the density as: $\rho = \dfrac{68.2\,\text{g}}{5\,\text{mL}} = 13.64$ g / mL.

1. Find the total differential $d\rho$ in terms of dm and dV.

Ans: $d\rho = \dfrac{1}{V}\,dm - \dfrac{m}{V^2}\,dV$

2. $d\rho$ helps us estimate the **propagated error** in using our measurements of mass and volume, which have uncertainties, to compute density. Find $d\rho$ for $m = 68.2$ g, $dm = \pm 0.1$ g, $V = 5$ mL and $dV = \pm 0.5$ mL:

Ans: Conservative error estimate: ± 1.384.

3. Find and interpret $\dfrac{d\rho}{\rho}$.

Ans: $\dfrac{d\rho}{\rho} = \pm \dfrac{1.384}{13.64} \approx \pm 10\,\%$. This means there is up to approximately $10\,\%$ relative error.

EXTENSIONS TO MORE VARIABLES:

Suppose $w = F(x, y, z)$ is a function of three variables.

1. What does it mean for F to be differentiable at a point (a, b, c)?

2. What would be the formula for the linearization of F at (a, b, c), $L_{(a,b,c)}(x, y, z)$?

3. How would you define the total differential dw?

EXAMPLE 6: To calculate the volume of a box, a student measures its dimensions with a ruler which is accurate to within an eighth of an inch. They find the length is 5 inches, the width is 4 inches, and the height is 2 inches. Use differentials to estimate the percent relative error in calculating the volume to be 40 cubic inches.

Ans: Using $V = \ell wh$, we get $dV = \pm 4.75$ cubic inches so $\dfrac{dV}{V} = \pm\dfrac{4.75}{40} \approx \pm 11.9\,\%$

PROVING A FUNCTION IS DIFFERENTIABLE USING THE DEFINITION:

Suppose we wish to prove the function $f(x, y) = xy^2$ is differentiable for all points (a, b) in the plane using the definition of differentiability. We need to find functions ϵ_1 and ϵ_2 so that

$$f(a + \Delta x, b + \Delta y) = f(a, b) + f_x(a, b)\Delta x + f_y(a, b)\Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y,$$

or

$$f(a + \Delta x, b + \Delta y) - [f(a, b) + f_x(a, b)\Delta x + f_y(a, b)\Delta y] = \epsilon_1 \Delta x + \epsilon_2 \Delta y,$$

with $\epsilon_1 \to 0$ and $\epsilon_2 \to 0$ as $\Delta x \to 0$ and $\Delta y \to 0$.

First we compute all the ingredients:

- $f(a + \Delta x, b + \Delta y) = (a + \Delta x)(b + \Delta y)^2 = ab^2 + 2ab\Delta y + a(\Delta y)^2 + b^2\Delta x + 2b\Delta x\Delta y + \Delta x(\Delta y)^2$
- $f_x(x, y) = y^2,\ f_y(x, y) = 2xy.$
- $f(a, b) = ab^2,\ f_x(a, b) = b^2,\ f_y(a, b) = 2ab.$
- $f(a, b) + f_x(a, b)\Delta x + f_y(a, b)\Delta y = ab^2 + b^2\Delta x + 2ab\Delta y$

Next, we subtract:

$$f(a + \Delta x, b + \Delta y) - [f(a, b) + f_x(a, b)\Delta x + f_y(a, b)\Delta y] = a(\Delta y)^2 + 2b\Delta x\Delta y + \Delta x(\Delta y)^2.$$

Now we sift through the leftovers to find candidates for ϵ_1 and ϵ_2:

Our goal is to rewrite $a(\Delta y)^2 + 2b\Delta x\Delta y + \Delta x(\Delta y)^2 = (\text{something})\Delta x + (\text{something else})\Delta y$ where the 'something' (i.e., ϵ_1) and 'something else' (i.e., ϵ_2) approach 0 as $\Delta x \to 0$ and $\Delta y \to 0$. One way is:[1]

$$a(\Delta y)^2 + 2b\Delta x\Delta y + \Delta x(\Delta y)^2 = (2b\Delta y + (\Delta y)^2)\Delta x + (a\Delta y)\Delta y$$

Viewed this way, we can take $\epsilon_1 = 2b\Delta y + (\Delta y)^2$ and $\epsilon_2 = a\Delta y$.

Both ϵ_1 and ϵ_2 go to 0 as $\Delta x \to 0$ and $\Delta y \to 0$.

Hence, $f(x, y) = xy^2$ satisfies the definition of differentiability for all points (a, b) in the plane. (WHEW!)

HOMEWORK: Section 15.6: 17, 19, 25 - 51 odd, 65*

[1] Can you find other choices for ϵ_1 and ϵ_2?